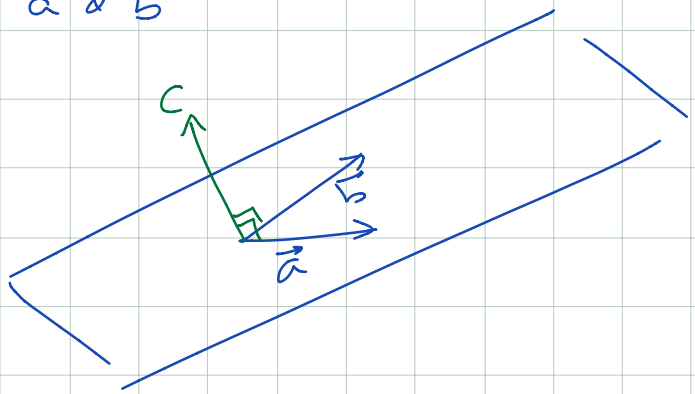
1.3) Matrices, Determinants and cross-product

Goal: Define the cross product $\vec{a} \times \vec{b} = \vec{c}$

where $\vec{c}$ is orthogonal to the plane spanned by $\vec{a}$ & $\vec{b}$



But First, we need a few things.

Matrices: a 2x2 matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

a 3x3 matrix $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

e.g. $\begin{bmatrix} 1 & 5 \\ -3 & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ 4 & 5 & -1 \end{bmatrix}$

The determinant of a 2×2 matrix is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

e.g. $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1\times 1 - 0\times 0 = 1$

$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\times 4 - 2\times 3 = -8$

The determinant of a 3×3 matrix is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

e.g.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\cdot\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= -3 - 2(-6) + 3(-3) = -3 + 12 - 9$$

$$= 0$$

Properties of Determinants

- Interchanging two rows or two columns results in a change of sign.

e.g: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = - \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix}$ (interchanging rows)

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 2 \\ 3 & 5 & 4 \\ 6 & 8 & 7 \end{vmatrix}$$ (interchanging columns)

exercise: Check to see this is true.

- We can factor a scalar out of any row or column, i.e.,

$$\begin{vmatrix} \alpha a_{11} & a_{12} \\ \alpha a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & \alpha a_{12} \\ a_{21} & \alpha a_{22} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= \begin{vmatrix} \alpha a_{11} & \alpha a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{vmatrix}$$

(Same for 3x3 matrices).

- adding a row (or column) from the matrix to another row (or column) from the matrix does not change the determinant.

e.g.: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+c & b+d \\ c & d \end{vmatrix} = \begin{vmatrix} a+b & b \\ c+d & d \end{vmatrix}$

Cross Products:

Suppose that $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

& $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

The cross product (or vector product) of $\vec{a}$ & $\vec{b}$ is $\vec{a} \times \vec{b}$, where the vector

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example: $(3\vec{i} - \vec{j} + \vec{k}) \times (\vec{i} + 2\vec{j} - \vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = \vec{i}(1-2) - \vec{j}(-3-1) + \vec{k}(6+1)$$

$$= -\vec{i} + 4\vec{j} + 7\vec{k}$$

Properties of cross products

(i) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

In particular, this implies

$\vec{a} \times \vec{a} = -\vec{a} \times \vec{a} \Rightarrow \vec{a} \times \vec{a} = 0$

(ii) $\vec{a} \times (\beta\vec{b} + \gamma\vec{c}) = \beta(\vec{a} \times \vec{b}) + \gamma(\vec{a} \times \vec{c})$

$(\alpha\vec{a} + \beta\vec{b}) \times \vec{c} = \alpha(\vec{a} \times \vec{c}) + \beta(\vec{b} \times \vec{c})$

By Property (i), we have $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$

Also, $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, $\vec{k} \times \vec{i} = \vec{j}$

From the definitions of cross product & scalar product, we have

$$\underbrace{(\vec{a}\times\vec{b})\cdot\vec{c}}_{\text{triple product}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \cdot (c_1\vec{i} + c_2\vec{j} + c_3\vec{k})$$

$$= \cdots = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Suppose that $\vec{c}$ lies in the plane spanned by $\vec{a}$ & $\vec{b}$ then $\vec{c} = \alpha\vec{a} + \beta\vec{b}$ for some α & β.

$$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{a}\times\vec{b})\cdot(\alpha\vec{a}+\beta\vec{b})$$

$$= (\vec{a}\times\vec{b})\cdot\alpha\vec{a} + (\vec{a}\times\vec{b})\cdot\beta\vec{b}$$

$$= \alpha\underbrace{(\vec{a}\times\vec{b})\cdot\vec{a}}_{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{vmatrix}} + \beta\underbrace{(\vec{a}\times\vec{b})\cdot\vec{b}}_{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 \end{vmatrix}}$$

check this $\rightarrow$ 0 check this $\rightarrow$ 0

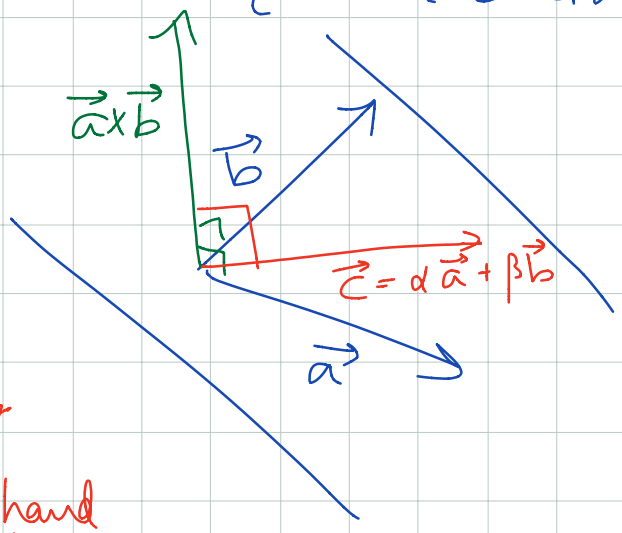
$$= 0$$

That is, $(\vec{a}\times\vec{b})\cdot\vec{c}=0$ in this case

So $\vec{a}\times\vec{b}$ is orthogonal to $\vec{a}$ & $\vec{b}$

and to all vectors $\vec{c}$ spanned by $\vec{a}$ & $\vec{b}$

(i.e. all $\vec{c}=\alpha\vec{a}+\beta\vec{b}$)



(Right hand rule)

- We have figured out the direction of $\vec{a}\times\vec{b}$
- Let us now figure out its length

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\ a_1&a_2&a_3\\ b_1&b_2&b_3\end{vmatrix}=\vec{i}\begin{vmatrix}a_2&a_3\\ b_2&b_3\end{vmatrix}-\vec{j}\begin{vmatrix}a_1&a_3\\ b_1&b_3\end{vmatrix}+\vec{k}\begin{vmatrix}a_1&a_2\\ b_1&b_2\end{vmatrix}$$

$$\Rightarrow \|\vec{a}\times\vec{b}\|^2=\begin{vmatrix}a_2&a_3\\ b_2&b_3\end{vmatrix}^2+\begin{vmatrix}a_1&a_3\\ b_1&b_3\end{vmatrix}^2+\begin{vmatrix}a_1&a_2\\ b_1&b_2\end{vmatrix}^2$$

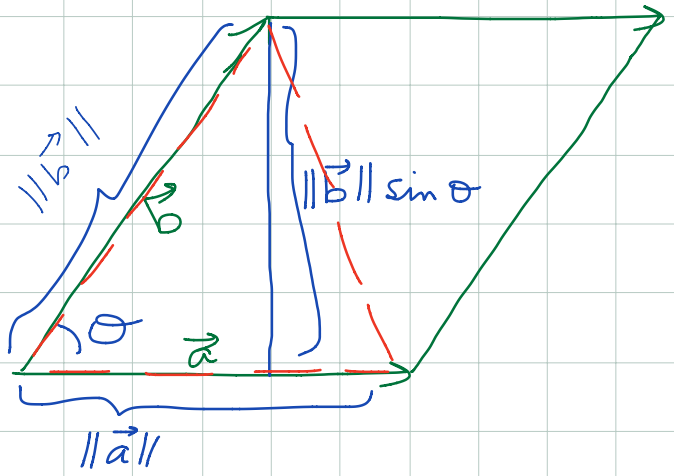
$$= \cdots$$

$$= \underbrace{(a_1^2 + a_2^2 + a_3^2)}_{\|\vec{a}\|^2}\underbrace{(b_1^2 + b_2^2 + b_3^2)}_{\|\vec{b}\|^2} - \underbrace{(a_1b_1 + a_2b_2 + a_3b_3)^2}_{(\vec{a}\cdot\vec{b})^2 = \|\vec{a}\|^2\|\vec{b}\|^2\cos^2\theta}$$

$$= \|\vec{a}\|^2\|\vec{b}\|^2(1 - \cos^2\theta) = \|\vec{a}\|^2\|\vec{b}\|^2\sin^2\theta$$

So: $$\boxed{\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\||\sin\theta|}$$

length of $\vec{a}\times\vec{b}$ = area of parallelogram



Consequence: $$\|\vec{a}\times\vec{b}\|^2 + (\vec{a}\cdot\vec{b})^2 = \|\vec{a}\|^2\|\vec{b}\|^2$$

So: $\vec{a}\times\vec{b}$ is $\perp$ to $\vec{a}$ & $\vec{b}$ (Right hand rule)

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\||\sin\theta|$$

Example: Find the area of the parallelogram spanned by $\vec{a} = (1,2,3)$, $\vec{b} = (1,1,1)$

Soln: Area $= \|\vec{a}\times\vec{b}\|$ where

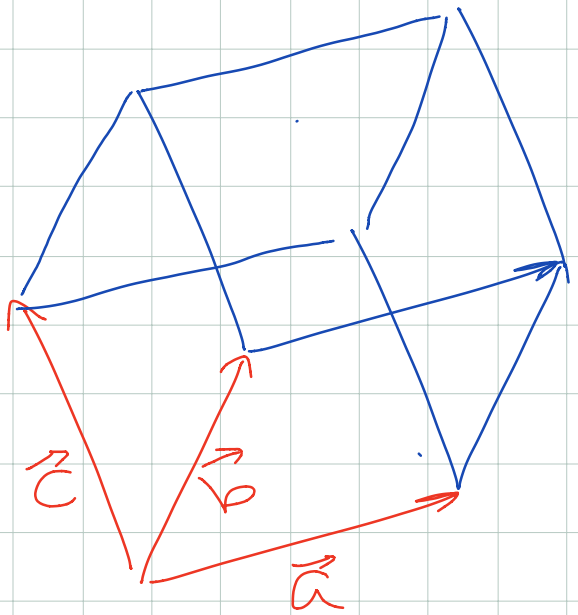
$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -\vec{i} + 2\vec{j} - \vec{k}$$

so Area $= \sqrt{1+4+1} = \sqrt{6}$

Example: Find a unit vector orthogonal to $\vec{a} = (1,2,3)$ & $\vec{b} = (1,1,1)$

Soln: the vector $\dfrac{\vec{a}\times\vec{b}}{\|\vec{a}\times\vec{b}\|}$ is orth to $\vec{a}$ & $\vec{b}$ and is of unit length. By the example above (same $\vec{a}$ & $\vec{b}$), we have $\dfrac{\vec{a}\times\vec{b}}{\|\vec{a}\times\vec{b}\|} = -\frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k}$

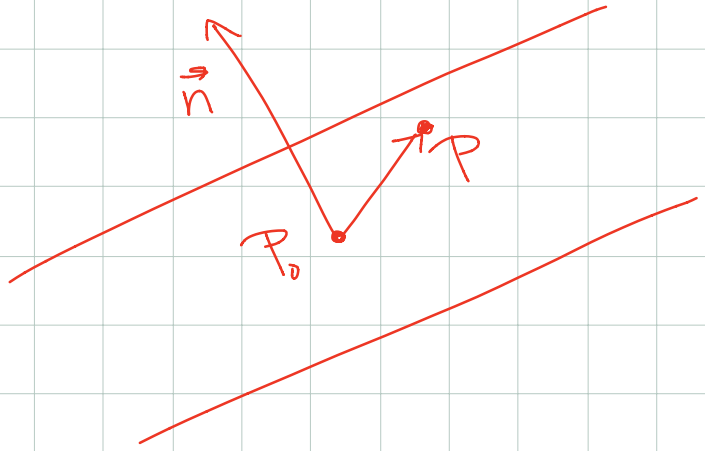
Remark: $|(\vec{a}\times\vec{b})\cdot\vec{c}|$ = volume (parallelpiped spanned by $\vec{a},\vec{b},\vec{c}$)



Example: Find the volume of the parallelpiped spanned by $\vec{a} = \vec{i} + 3\vec{k}$, $\vec{b} = 2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{c} = 5\vec{i} + 4\vec{k}$

Sol'n: $(\vec{a}\times\vec{b})\cdot\vec{c} = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 1 & -2 \\ 5 & 0 & 4 \end{vmatrix} = -11 \Rightarrow \text{Vol} = 11.$

Equations of planes



Let $\vec{n} = A\vec{i} + B\vec{j} + C\vec{k}$ be normal to the plane

So $\vec{n}$ is orthogonal to all vectors in the plane

So, if $P_0 = (x_0, y_0, z_0)$ & $P = (x, y, z)$ are two points

on the plane, $\vec{n} \cdot \overrightarrow{P_0P} = 0$

i.e.

$$(A\vec{i} + B\vec{j} + C\vec{k}) \cdot \left((x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}\right) = 0$$

So $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Eq'n of plane normal to $\vec{n} = (A, B, C)$
and passing thru $P_0 = (x_0, y_0, z_0)$
& $P = (x, y, z)$

So a point $P=(x,y,z)$ belongs to the plane if $Ax+By+Cz+D=0$ where $D=-Ax_0-By_0-Cz_0$

Example: Find the equation of the plane containing the three points $\underbrace{(1,1,1)}_{P}, \underbrace{(2,0,0)}_{Q}, \underbrace{(1,1,0)}_{R}$

Soln: Need to find a normal vector to the plane

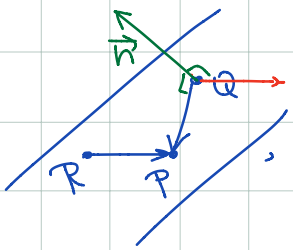
For example $\vec{n} = \overrightarrow{QP} \times \overrightarrow{RP} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}+\vec{j}$

So the eqn of the plane is

$$\overset{1}{\searrow} A(x-x_0) + \overset{1}{\searrow} B(y-y_0) + \overset{0}{\searrow} C(z-z_0) = 0$$

$x_0 = 2$, $y_0 = 0$, $z_0 = 0$

i.e. $(x-2)+y=0 \iff \boxed{x+y=2}$



Two Planes are Parallel when their normal vectors are parallel

So $A_1x + B_1y + C_1z + D_1 = 0$

And

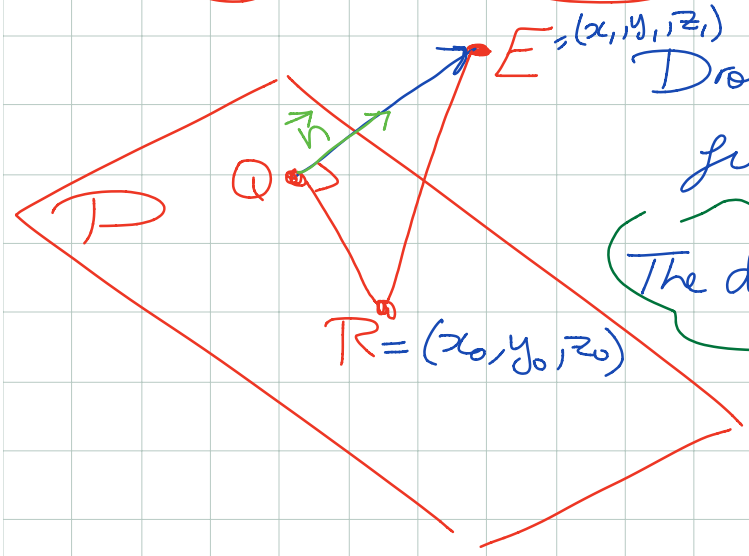
$A_2x + B_2y + C_2z + D_2 = 0$

are parallel if $(A_1, B_1, C_1) = \alpha(A_2, B_2, C_2)$

for some $\alpha \neq 0$

e.g. The planes given by $2x + y - 3z + 5 = 0$ & $4x + 2y - 6z + 107 = 0$ are parallel.

Distance from A point to a plane



Drop a perpindicular from E to the plane P.

The distance from E to P is $\|\vec{QE}\|$

Let $\vec{n}$ be the unit normal to the plane P given by $Ax + By + Cz + D = 0$

so $\vec{n} = \dfrac{A\vec{i} + B\vec{j} + C\vec{k}}{\sqrt{A^2+B^2+C^2}}$

From the picture: distance = length of proj. of $\vec{RE}$ onto $\vec{n}$ ...

$$\Rightarrow \quad \text{distance} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2+B^2+C^2}}$$

Example: Find the distance of the point $(1,1,1)$ to the plane $2x + 3y - 4z + 1 = 0$

Sol'n: $\text{dist} = \dfrac{2(1) + 3(1) - 4(1) + 1}{\sqrt{2^2+3^2+4^2}} = \dfrac{2}{\sqrt{29}}$